Midterm Exam: MAT 362

Instructions: Complete all of the problems 1-5. You may not use calculators or any other electronic devices, books or notes (except the cheat sheet). In order to receive credit you must show all of your work and justify each statement. **Be sure to write your name and student ID on each page that you hand in.**

1. (20 points) Consider the helicoid $X(u, v) = (v \cos u, v \sin u, v)$, $0 < u < 2\pi$, $-\infty < v < \infty$. Calculate its first fundamental form.

$$\underline{X}_u = (-v\sin u, v\cos u, 0)$$

$$\underline{X}_v = (\cos u, \sin u, 1).$$

$$\Rightarrow E = \underline{X}_u \cdot \underline{X}_u = v^2$$

$$F = \underline{X}_u \cdot \underline{X}_v = 0$$

$$G = \underline{X}_v \cdot \underline{X}_v = 2$$

Thus the first fundamental form is.

$$I = v^2 du^2 + 2 dv^2.$$

2. (20 points) Let $\alpha : (-1, 1) \to \mathbb{R}^3$ be a regular curve. If the torsion $\tau = 0$, show that α must lie in a plane.

$T(s) = \alpha'(s)$, $\quad N(s) = \dfrac{\alpha''(s)}{k(s)}$, $\quad B = T \times N$.

$T' = kN$

$N' = -kT + \tau B$.

$B' = -\tau N$.

$\tau = 0 \Rightarrow B(s) = \text{const}$.

$(\alpha \cdot B_0)' = \alpha' \cdot B_0 = T \cdot B_0 = 0$.

Hence, for all $t \in (-1,1)$ we have

$\alpha(t) \cdot B_0 = C$ for some constant C.

This is exactly the equation for a plane,

It follows that α must lie in a plane.

3. (20 points) Show that the ellipsoid ($a \neq 0,\ b \neq 0,\ c \neq 0$)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface.

Consider $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 .$$

$f^{-1}(0) =$ ellipse.

0 is a regular value because

$\nabla f = \left(\frac{2x}{a^2} , \frac{2y}{b^2} , \frac{2z}{c^2} \right) = 0 \Rightarrow (x, y, z) = 0$

and this point is not in $f^{-1}(0)$.

4. (20 points) Let S^2 be the unit sphere in $\mathbb{R}^3$, and let $A : S^2 \to S^2$ be the antipodal map given by $A(p) = -p$. Prove that A is a diffeomorphism.

Surjective: Given $p \in S^2$ $\exists$ $-p \in S^2$ such that $A(-p) = -(-p) = p$.

Injective: If $A(p) = A(q)$ for $p, q \in S^2$ then $-p = -q \Rightarrow p = q$.

Smoothness: Let $\tilde{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\tilde{A}(x, y, z) = (-x, -y, -z)$. Then $A = \tilde{A}\big|_{S^2}$. Clearly $\tilde{A}$ is smooth. Thus, A is the restriction to a regular surface S^2 of a smooth map between $\mathbb{R}^3$. Hence by Example 3 on page 77 of the book A is smooth as a map from $S^2 \to S^2$.

Smoothness of Inverse: Since $A^{-1} = A$ we get that A^{-1} is smooth.

All these properties together imply that A is a diffeo.

5. (20 points) Let $X(u,v) = ((v^2+1)\cos u, \sin u, v)$, $0 < u < 2\pi$, $-\infty < v < \infty$ be the parameterization of a regular surface S. Find a basis $\{w_1, w_2\}$ for the tangent plane T_pS at any point $p \in S$. Consider the function $f : S \to \mathbb{R}$ given by $f(x,y,z) = xz$. Find the images of w_1, w_2 under the differential $df_p : T_pS \to T_{f(p)}\mathbb{R}$.

$$X_u = (-(v^2+1)\sin u, \cos u, 0)$$

$$X_v = (2v\cos u, 0, 1).$$

Then $\{w_1 = X_u,\ w_2 = X_v\}$ forms a basis for T_pS at any point $p = X(u,v)$.

$$\alpha(t) = X(u(t), v(t)) = X(t+u_0, v_0).$$

$$\beta(t) = X(u(t), v(t)) = X(u_0, t+v_0)$$

Let $p = X(u_0, v_0)$.

$$df_p(w_1) = \frac{d}{dt} f\circ\alpha(t)\Big|_{t=0} = \frac{d}{dt} \underbrace{(v_0^2+1)\cos(t+u_0)}_{x} \cdot \underbrace{v_0}_{z}\Big|_{t=0}$$

$$= -v_0(v_0^2+1)\sin(t+u_0)\Big|_{t=0} = \boxed{-v_0(v_0^2+1)\sin u_0}$$

$$df_p(w_2) = \frac{d}{dt} f\circ\beta(t)\Big|_{t=0} = \frac{d}{dt}\left((t+v_0)^2+1\right)\cos u_0 \cdot (t+v_0)\Big|_{t=0}$$

$$= \left[\left((t+v_0)^2+1\right)\cos u_0 + 2(t+v_0)^2\cos u_0\right]\Big|_{t=0}$$

$$= \left((v_0^2+1) + 2v_0^2\right)\cos u_0$$

$$= \boxed{(3v_0^2+1)\cos u_0}$$